Interpolating Scattered Data With C^2 Surfaces

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Abstract

A method is presented for interpolating bivariate scattered data based upon a minimum norm network. This new method is related to G. M. Nielson's minimum norm network and to H. Pottmann's generalization of that method. The shape of the resulting C^2 interpolant is controlled parametrically. Examples are given showing how the surface responds to changes in the control parameter. To illustrate the results, curvature plots as well as shaded images of the interpolants are given.

Introduction

The problem of interpolating scattered data has been investigated by many authors in recent years (for an overview: cf. [1], [3], [5] and [8]). In 1983, Nielson introduced a method for solving this problem based on variational principles, which leads to the minimum norm network (MNN) (cf. [14]). This approach was later extended using splines under tension (instead of cubic splines) as basis functions on the edges of a given domain triangulation (cf. [15]). Both methods yield a C^1 interpolant. Pottmann developed a method for interpolating scattered data with C^r surfaces (arbitrary r) using a variational principle. The solution gives not only the function values of the network, but also values of cross-boundary derivatives (cf. [17]).

Assuming that data points $V_i = (x_i, y_i) \in \Omega \subseteq \mathbb{R}^2, \quad i \in \{1, \ldots, n\}$, and reals $z_i, \quad i = 1, \ldots, n$ are given, all of the methods mentioned above consist of three steps:

Step 1: The data points are used to construct a *triangulation* of the domain Ω.

Step 2: A specific variational problem is introduced, whose solution is an interpolating *curve network* defined over the edges of the triangulation.

Step 3: The network is extended to the interior of the triangles by means of a *triangular interpolant.*

In general, it is desirable to have a surface with continuity more than C^1 and a means of controlling the geometric properties of that surface. Another point of interest is the quality of the interpolating surface, e.g. we desire a surface with uniform curvature distribution, upon which the patch structure of the surface has no disturbing effects. The new variational principle used in this paper has been chosen with this in mind.

Variational Principles and Minimum Norm Networks

Consider a set of non-collinear vertices $V_i \in \Omega \subseteq \mathbb{R}^2,\ i = 1, \ldots, n$. We do not deal with the question of how to triangulate the domain Ω. Rather we will take as input a given triangulation (for methods to construct a triangulation see e.g. [4], [6], [7] and [18]).

In the given triangulation T, we let e_{ij} be the edge between V_i and V_j. We refer to the corresponding vector of unit length by $\underline{e}_{ij}$, i.e., $\underline{e}_{ij} = (V_j - V_i)/\|e_{ij}\|$, where $\|e_{ij}\| = \|V_j - V_i\|$. The set $E = \{ij : e_{ij} \text{ is an edge of T}, i < j\}$ contains one double index for each edge. Furthermore, let $E_i = \{j : e_{ij} \text{ is an edge of T}\}$. The domain of the curve network is the union of all edges

$$\mathcal{E} = \bigcup_{ij \in E} e_{ij}.$$

In addition, for a vector $\underline{v} \in \mathbb{R}^2$ of unit length we denote the k-th directional derivative of a bivariate function F in direction of $\underline{v}$ by $\frac{\partial^k F}{\partial \underline{v}^k}$.

The approach used by Nielson and Franke (cf. [15]) is to minimize

$$\sigma(F) = \sum_{ij \in E} \int_{e_{ij}} \left\{ \left(\frac{\partial^2 F}{\partial \underline{e}_{ij}^2} \right)^2 + \alpha_{ij}^2 \left(\frac{\partial F}{\partial \underline{e}_{ij}} \right)^2 \right\} de_{ij} \tag{1}$$

where de_{ij} is the element of arc length along e_{ij}. The unique solution to this minimization problem is a piecewise spline under tension, or a piecewise cubic polynomial, in case of $\alpha_{ij} \neq 0$ or $\alpha_{ij} = 0$, respectively. The parameters α_{ij} are called *tension parameters* on the edge e_{ij}. Increasing α_{ij} results in gradual linearization of the network over the corresponding edge e_{ij}.

Pottmann's generalized minimum norm network (cf. [17]) is obtain by minimizing

$$\sigma^r(F) = \sum_{ij \in E} \int_{e_{ij}} \left[\sum_{k=0}^{r} \beta_k \left(\frac{\partial^{r+1} F}{\partial \underline{n}_{ij}^k \partial \underline{e}_{ij}^{r-k+1}} \right)^2 \right] de_{ij}, \quad \beta_k \geq 0, \quad k = 0, \ldots, r \tag{2}$$

for some r. Here $\underline{n}_{ij}$ denotes a vector of unit length that is normal to $\underline{e}_{ij}$. Introducing cross boundary derivatives into the functional gives information about the form of the cross boundary derivatives of the unique minimizer along edges. The solution is piecewise polynomial of degree $2r + 1$ with k-th order normal derivatives of degree $2(r - k) + 1$. The parameters β_k control the relation between the different derivatives of the same order. Their geometric effects on the network are difficult to predict. Note that when $r = 1$, the network can not be put under tension, because first order directional derivatives are not considered in (2).

To formulate the new variational principle, we first define an appropriate set of univariate functions over an interval I:

$$\mathcal{H}(I) = \{f : f'' \text{ absolutely continuous, } f^{(3)} \in \mathcal{L}_2(I)\},$$

and the corresponding set of network functions

$$\mathcal{H}^*(\mathcal{E}) = \left\{ F : F = G\Big|_{\mathcal{E}} \text{ where } G \in C^2(\Omega),\ F\Big|_{e_{ij}} \in \mathcal{H}(e_{ij}) \right\}. \tag{3}$$

We now consider the functional

$$\sigma^*(F) = \sum_{ij\in E} \int_{e_{ij}} \left\{ \left(\frac{\partial^3 F}{\partial \underline{e}_{ij}^3} \right)^2 + \gamma_{ij}^2 \left(\frac{\partial^2 F}{\partial \underline{e}_{ij}^2} \right)^2 \right\} de_{ij}, \quad \gamma_{ij} \neq 0, \ \forall ij \in E. \tag{4}$$

We will see that the solution to the corresponding minimization problem can be represented locally in the form

$$s(u) = a + bu + cu^2 + du^3 + le^{\gamma u} + \mu e^{-\gamma u}. \tag{5}$$

This type of function is known as a *generalized spline under tension* (for more information about this type of spline, see [16]). For a given curve network $F \in \mathcal{H}^*(\mathcal{E})$, we define the univariate function f_{ij} over the edge e_{ij} to be

$$f_{ij}(t) = F((1-t)V_i + tV_j), \quad t \in [0,1].$$

Before we can state our main result, we introduce local coordinate systems for each vertex $V_i,\ i = 1, \ldots, n$. Since we will have to deal with first and second order directional derivatives, we have to distinguish between two kinds of vertices. If we find at least three pairwise linearly independent vectors $\underline{e}_{ij}$ for $j \in E_i$, we use ordinary Cartesian coordinates (we say, V_i satisfies the *three-edge* condition). Due to $F \in \mathcal{H}^*(\mathcal{E})$, there exist uniquely determined first and second order partial derivatives at these vertices. In case we find just two pairwise linear independent vectors, second order partial derivatives are not unique. In this situation we choose two linear independent vectors of unit length $\underline{e}_{ij_1},\ \underline{e}_{ij_2},\ j_1, j_2 \in E_i$ as base for the coordinate system at V_i. We use the following notations: Δu_{ij} and Δv_{ij} denote the coordinates of $V_j - V_i$ with respect to the local coordinate system at V_i. $F_u(V_i), F_v(V_i)$ are partial derivatives of F at V_i also with respect to the local base $\{u, v\}$ at V_i.

Theorem 1. *Assume that $V_i,\ i \leq k$ for some $k \leq n$ satisfy the three-edge condition, whereas $V_i,\ i > k$ do not. Then there exists a unique minimizer S of σ^* in $\mathcal{H}^*(\mathcal{E})$ subject to $F(V_i) = z_i$. S is a piecewise generalized spline under tension network, i.e.,*

$$s_{ij}(u) = a_{ij} + b_{ij}u + c_{ij}u^2 + d_{ij}u^3 + l_{ij}e^{\gamma_{ij}\|e_{ij}\|u} + \mu_{ij}e^{-\gamma_{ij}\|e_{ij}\|u},$$

and satisfies the properties

$$\begin{aligned}
\sum_{j\in E_i} \frac{\Delta u_{ij}}{\|e_{ij}\|^5} \left(s_{ij}^{(4)}(0) - \gamma_{ij}^2 \|e_{ij}\|^2 s_{ij}''(0) \right) &= 0 \\
\sum_{j\in E_i} \frac{\Delta v_{ij}}{\|e_{ij}\|^5} \left(s_{ij}^{(4)}(0) - \gamma_{ij}^2 \|e_{ij}\|^2 s_{ij}''(0) \right) &= 0 \\
\sum_{j\in E_i} \frac{(\Delta u_{ij})^2}{\|e_{ij}\|^5} s_{ij}^{(3)}(0) &= 0 \\
\sum_{j\in E_i} \frac{(\Delta v_{ij})^2}{\|e_{ij}\|^5} s_{ij}^{(3)}(0) &= 0 \\
& \quad i = 1, \ldots, n,
\end{aligned} \tag{6}$$

and

$$\begin{aligned}
\sum_{j\in E_i} \frac{(\Delta u_{ij})(\Delta v_{ij})}{\|e_{ij}\|^5} s_{ij}^{(3)}(0) &= 0 \\
& \quad i = 1, \ldots, k.
\end{aligned}$$

(Note: if V_i does not satisfy the three-edge condition, either $\Delta u_{ij} = 0$ or $\Delta v_{ij} = 0 \; \forall j \in E_i$.)

Proof. Since we use standard arguments for minimum norm network theorems, we will not go into many details here (more detailed proofs can be found in [14] and [15]).

First, let us define the inner product for $F, G \in \mathcal{H}^*(\mathcal{E})$ associated with σ^*

$$\langle \; F \mid G \; \rangle = \sum_{ij \in E} \int_{e_{ij}} \left\{ \frac{\partial^3 F}{\partial \underline{e}_{ij}^3} \frac{\partial^3 G}{\partial \underline{e}_{ij}^3} + \gamma_{ij}^2 \frac{\partial^2 F}{\partial \underline{e}_{ij}^2} \frac{\partial^2 G}{\partial \underline{e}_{ij}^2} \right\} \, de_{ij}.$$

We now assume that there exists a piecewise generalized spline under tension network satisfying the interpolation condition and the linear system (6).

For any $H \in \mathcal{H}^*(\mathcal{E})$ subject to $H(V_i) = Z_i, i = 1, \ldots, n$, we have $\sigma^*(H) - \sigma^*(S) = \sigma^*(H-S) + 2\langle \; S \mid H-S \; \rangle$. Thus, the minimum property can be established by proving $\langle \; S \mid H-S \; \rangle = 0$.

By introducing s_{ij} and h_{ij} and using integration by parts, the interpolation conditions and the fact that s_{ij} satisfies the integral equation $y^{(6)} - (\gamma_{ij} \, \|e_{ij}\|)^2 \, y^{(4)} = 0$, we get

$$\begin{aligned} \langle \, S \mid H - S \, \rangle \;=\; & \sum_{i=1}^{n} \sum_{j \in E_i} \frac{1}{\|e_{ij}\|^5} \Big\{ -s_{ij}^{(3)}(0) \left(h_{ij}''(0) - s_{ij}''(0) \right) \\ & - \left((\gamma_{ij} \, \|e_{ij}\|)^2 \, s_{ij}''(0) - s_{ij}^{(4)}(0) \right) \left(h_{ij}'(0) - s_{ij}'(0) \right) \Big\}, \end{aligned} \tag{7}$$

(note that $s_{ij}^{(k)}(t) = (-1)^k s_{ji}^{(k)}(1-t)$). Replacing $h_{ij}'(0) - s_{ij}'(0)$ and $h_{ij}''(0) - s_{ij}''(0)$ with the first and second order partial derivatives of $H - S$ (with respect to the local coordinate system) and rearranging (7) appropriately, we find the coefficients of $(H-S)_u, \ldots, (H-S)_{vv}$ to be the left sides of (6). We conclude $\langle \; S \mid H - S \; \rangle = 0$. Thus, if S exists, it has the minimum property.

To prove the existence and uniqueness of S, we have to examine the linear system (6). Using Hermite functions to express the generalized splines under tension s_{ij} and introducing partial derivatives (with respect to the local bases) yields a $(4n+k) \times (4n+k)$ system with the partial derivatives as unknowns. At vertices $V_i, \; i > k$, we have to determine only two first and two second order directional derivatives. Assuming that this system is not regular, we can find a nontrivial solution for the corresponding homogeneous system (i.e., $z_i = 0, \; i = 1, \ldots, n$). Let $\overline{S}$ denote the uniquely determined generalized spline under tension network associated with this solution. $H \equiv 0$ satisfies the interpolating conditions in the homogeneous case. Thus, we have $\langle \, \overline{S} \mid \overline{S} \, \rangle = \sigma^*(\overline{S}) = 0$. This implies that $\overline{S}$ is piecewise linear ($\gamma_{ij} \neq 0$) and since $z_i = 0$, we conclude $\overline{S} \equiv 0$. At vertices $V_i, \; i \leq k$ the two first and three second order partial derivatives vanish. For $i > k$ we consider only directional derivatives along edges, thus $\overline{S}_u(V_i) = \ldots = \overline{S}_{vv}(V_i) = 0, \; i = 1, \ldots, n$. Since this contradicts the non-triviality of the solution of the linear system assumed above, S exists and is unique.

■

If we take a closer look at the functional (1) for the case $\alpha_{ij} = 0, \; \forall ij \in E$, we find the following relationship between the solutions of the minimization problems related to the functionals σ (1) and σ^* (4).

Corollary 2 *Let S_γ denote the unique solution of the minimization problem connected with*

(4), where $\gamma = \gamma_{ij}$, $\forall ij \in \mathcal{E}$ is a global parameter. Then

$$S_\gamma \stackrel{\gamma\to\infty}{\longrightarrow} S,$$

uniformly, where S is the unique minimum of σ subject to the interpolation problem for $\alpha_{ij} = 0$, $\forall ij \in E$ (see (1)).

Proof. Let σ_γ^* be the functional (4) with the global parameter γ. Since S_γ uniquely minimizes σ_γ^*, we have $\sigma(S_\gamma) \le \frac{1}{\gamma^2}\sigma_\gamma^*(S_\gamma) < \frac{1}{\gamma^2}\sigma_\gamma^*(S)$.

We define $\langle\, . \mid . \,\rangle$ to be the inner product related with σ. Similarly to the proof of Theorem 1 it can be shown that $\langle\, S \mid S_\gamma - S \,\rangle = 0$. Using the minimum property of S for the functional σ, we can state that

$$\begin{aligned} \sigma(S_\gamma - S) &= \sigma(S_\gamma) - \sigma(S) + 2\langle\, S \mid S - S_\gamma \,\rangle \\ &< \frac{1}{\gamma^2}\sigma_\gamma^*(S) - \sigma(S) = \frac{1}{\gamma^2}\sum_{ij\in E}\int_{e_{ij}}\left(\frac{\partial^3 S}{\partial \underline{e}_{ij}^3}\right)^2 de_{ij}. \end{aligned} \tag{8}$$

Let us define univariate functions $f_{ij,\gamma}(u) = (S_\gamma - S)\left((1-u)V_i + uV_j\right)$. Then (8) implies $\int_0^1 \left(f''_{ij,\gamma}(u)\right)^2 du \to 0$, if $\gamma \to \infty$. $f_{ij,\gamma}$ can be expressed as

$$f_{ij,\gamma}(u) = (1-u)\, f_{ij,\gamma}(0) + u f_{ij,\gamma}(1) + \int_0^1 K(u,t) f''_{ij,\gamma}(t)\, dt,$$

where

$$K(u,t) = \begin{cases} t(u-1), & t \in [0,u) \\ u(t-1), & t \in [u,1] \end{cases}.$$

Together with the interpolation properties $f_{ij,\gamma}(0) = f_{ij,\gamma}(1) = 0$ and the Schwarz inequality for all $u \in [0,1]$ we get

$$|f_{ij,\gamma}(u)| \le \left(\int_0^1 (K(u,t))^2\, dt\right)^{\frac{1}{2}} \cdot \left(\int_0^1 \left(f''_{ij,\gamma}(t)\right)^2 dt\right)^{\frac{1}{2}}. \tag{9}$$

Since $K(u,t)$ is continuous, there exists a real M such that $\int_0^1 (K(u,t))^2\, dt < M$, $\forall u \in [0,1]$. This establishes the uniform convergence $S_\gamma \to S$. ■

Remark 1 The $(4n+k) \times (4n+k)$ linear system implicitly given by (6) can be solved with the *successive overrelaxation*-method *(SOR)*. This method is based on the iterative Gauss-Seidel-algorithm. In any case, our situation requires the use of a block matrix version of either method, involving explicit inversion of a series of block matrices (5×5, $i = 1, \ldots, k$ and 4×4, $i = k+1, \ldots, n$). The number of iteration steps can be reduced significantly in comparison with the ordinary Gauss-Seidel method (cf. [19]).

Remark 2 In Theorem 1 we prove the existence and uniqueness of the interpolating network in the case that the domain $\mathcal{E}$ of our network is the union of the edges of a given triangulation in the parameter space $\Omega \subseteq \mathbb{R}^2$. However it can be shown that we get the same results in the much more general situation

- $V_i \in \mathbb{R}^d$, $i = 1, \ldots, n$, for a fix $d \in \mathbb{N}$,
- $E \subseteq \{ij : i,j \in \{1,,\ldots,n\},\ i < j\}$ and $E_i = \{j : ij \text{ or } ji \text{ is in } E\}$, with $|E_i| > 0$ and
- $\mathcal{E}$ the union of all edges $\bigcup\{e_{ij} :\ \forall ij \in E\}$.

Extension of the Network

Since we want not only to extend the network, but also to transfer the geometric properties from the network to the resulting surface, we use side-vertex-interpolants on triangles (cf. [13]). These interpolants are based on univariate Hermite functions.

The usage of these interpolants requires complete information about first and second order partial derivatives over all three edges of the considered triangle. Since at vertices V_i, $i > k$, we have only two second order directional derivatives, the second partial derivatives are not uniquely determined. We assume that the two given directions are *conjugate directions*, which gives us a full characterization of the curvature and thus of the second partial derivative in these vertices (cf. [2]). To determine the cross boundary derivative $\frac{\partial S}{\partial \underline{n}_{ij}}$, we use the unique cubic polynomial that interpolates $\frac{\partial S}{\partial \underline{n}_{ij}}$ and $\frac{\partial^2 S}{\partial \underline{n}_{ij} \partial \underline{e}_{ij}} = \frac{\partial}{\partial \underline{e}_{ij}} \left(\frac{\partial S}{\partial \underline{n}_{ij}} \right)$ at the vertices V_i and V_j. By linearly interpolating the second directional derivates $\frac{\partial^2 S}{\partial \underline{n}_{ij}^2}$ we obtain complete information about all derivatives along the edge e_{ij}.

Consider the nondegenerate triangle T with vertices V_1, V_2, V_3. Every point $P \in \mathbb{R}^2$ can be uniquely expressed in barycentric coordinates b_1, b_2, b_3 with respect to V_1, V_2, V_3 (i.e., $P = b_1 V_1 + b_2 V_2 + b_3 V_3,\ b_1 + b_2 + b_3 = 1$). Let $\{i, j, k\} = \{1, 2, 3\}$. If $P \in T \backslash \{V_1, V_2, V_3\}$, we denote the point of intersection between e_{ij} and the line through P and V_k by $S_k(P)$. Let $h_{i,0}, h_{i,1} \in span\{1, t, t^2, t^3, e^{\delta t}, e^{-\delta t}\}$ be Hermite functions so that

$$h_{i,0}^{(k)}(1) = h_{i,1}^{(k)}(0) = 0 \quad , \quad h_{i,0}^{(k)}(0) = h_{i,1}^{(k)}(1) = \delta_{i,k}, \quad i, k \in \{0, 1, 2\}.$$

For the vertex V_1, we define the patch

$$D_1[S](P) = \sum_{i=0}^{2} \{ d_{i,0}(P) h_{i,0}(1 - b_1) + d_{i,1}(P) h_{i,1}(1 - b_1) \},$$

where $\Delta = \|S_1(P) - V_1\|$, $\underline{v} = (S_1(P) - V_1)/\Delta$ and:

$$\begin{aligned} d_{0,0}(P) &= S(V_1) \quad , \quad d_{0,1}(P) = S(S_1(P)), \\ d_{1,0}(P) &= \Delta \cdot \frac{\partial S}{\partial \underline{v}}(V_1) \quad , \quad d_{1,1}(P) = \Delta \cdot \frac{\partial S}{\partial \underline{v}}(S_1(P)), \\ d_{2,0}(P) &= \Delta^2 \cdot \frac{\partial^2 S}{\partial \underline{v}^2}(V_1) \quad , \quad d_{2,1}(P) = \Delta^2 \cdot \frac{\partial^2 S}{\partial \underline{v}^2}(S_1(P)). \end{aligned}$$

If we choose $\delta = \delta(P) = (b_2 \gamma_{12} + b_3 \gamma_{13}) / (b_2 + b_3)$, the patch $D_1[S]$ interpolates function values over all three edges, first and second order partial derivatives over e_{ij} as well as the first and second order directional derivatives at V_1 of the network S.

The patches $D_2[S]$ and $D_3[S]$ are defined analogously. The final patch is obtained by a convex combination of these three patches:

$$D[S] = \sum_{i=1}^{3} \omega_i D_i[S],$$

where ω_i are weight functions

$$\omega_i(P) = \frac{b_j^2 b_k^2}{b_1^2 b_2^2 + b_2^2 b_3^2 + b_1^2 b_3^2}.$$

This patch combines the interpolating properties of $D_1[S], D_2[S]$ and $D_3[S]$.

Examples

The presented scheme was designed for the construction of C^2 surfaces where the shape can be controlled parametrically and where the resulting surface has a uniform appearance, i.e., the use of triangular patches has no negative effects on the curvature distribution of the surface as a whole.

We will not deal with the problem of approximating test functions here. However, it should be pointed out that in various tests the approximation properties of the above described scheme turn out better than those produced by the MNN method (cf. [10]).

1 (a)

1 (b)

1 (c)

1 (d)

Figure 1(a): Nielson's MNN for $\alpha = 0$.
Figure 1(b)-(d): Interpolating C^2 surfaces for $\gamma = 0.1, 1.0, 10.0$

In our first example we use a Delaunay triangulation of the data points taken from ([15], p 210) and data values corresponding to these data points. Here we want to demonstrate the effect that the parameter γ has on the shape of the interpolant. Figure 1(a) shows the MNN interpolant for $\alpha = 0$, whereas Figures 1(b)-(c) show the C^2 surface with parameters $\gamma =$

$0.1, 1.0$ and 10.0. As we know from Corollary 2, the network converges to a piecewise cubic network as we increase γ. Due to the choice of our triangular interpolant, the whole surface behaves in the same way. This gives a shape for large parameters that is approximately the C^1 surface resulting from the MNN but is still a C^2 surface.

For our next example we take a rectangular grid of 6×6 data points. The data values are sampled from the test function f (taken from [9]):

$$f(u,v) = \left(1 - \frac{u}{2}\right)^6 \left(1 - \frac{v}{2}\right)^6 + 1000\,(1-u)^3\, u^3\, (1-v)^3\, v^3 + v^6 \left(1 - \frac{u}{2}\right)^6 + u^6 \left(1 - \frac{v}{2}\right)^6 .$$

2 (a)

2 (b)

2 (c)

Figure 2(a): Curvature plots of the test function.
Figure 2(b),(c): Curvature plots of the preliminary implementation (2(b)) and of the improved implementation as described in this paper (2(c)).

To illustrate the curvature distribution, we use color curvature plots to display *maximum* curvature. Here areas of strong positive curvature are blue, whereas green indicates regions of strong negative curvature. The color shifts from blue through magenta, red and yellow corresponding to curvature changes from positive to negative.

Figure 2(a) shows a curvature plot of the test function.

In a first implementation of the scheme, a somewhat different method of extending normal derivatives to the boundaries of the triangles was used. Although the resulting surface had a pleasing shape, curvature plots showed a quite non-uniform distribution of curvature (see figure 2(b)). Here the patch structure is clearly visible. The improved scheme, as presented in this paper, turn out to have a much more uniform curvature distribution, and, above all, the patch structure is almost invisible in the resulting curvature plots (see figure 2(c)).

Finally, we want to state explicitly that the visual appearance and the curvature distribution of the resulting surface is also smooth around vertices which do not satisfy the three-edge condition.

Future Investigations

In this paper we discussed the case of functional surfaces. There are ways to extend Nielson's MNN to the case of parametric interpolation (cf. e.g. [8]). Since Nielson's method yields a C^1 surface, methods derived from that lead to G^1 surfaces. However, Lounsbery, Mann and DeRose [8] observed that the resulting surface tends to have a very non-uniform curvature distribution. They conclude that the use of convex combination schemes for the extension of the network is not advisable. Moreton and Séquin use methods similar to Nielson's MNN to construct a G^2 network in the parametric interpolation case. However, they use Bézier patches to extend the network to a (approximately) G^1 surface (cf. [11] and [12]). It is hoped that the extension to the parametric case of the C^2 scheme presented in this paper will lead to a local G^2 scheme with improved curvature distribution.

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